

# A PROBABILISTIC INTERPRETATION OF THE VOLKENBORN INTEGRAL

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**ABSTRACT.** In this paper, we provide a probabilistic interpretation of the Volkenborn integral; this allows us to extend results by T. Kim et al about sums of Euler numbers to sums of Bernoulli numbers. We also obtain a probabilistic representation of the multidimensional Volkenborn integral which allows us to derive a multivariate version of Raabe's multiplication theorem for the higher-order Bernoulli and Euler polynomials.

## 1. INTRODUCTION

The Volkenborn integral was introduced in 1971 by A. Volkenborn in his PhD dissertation and subsequently in the set of twin papers [4]; a more recent treatment of the subject can be found in [5]. The Volkenborn integral, or fermionic  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$ , of a function  $f$  is defined as

$$\int_{\mathbb{Z}_p} f(y) d\mu_{-q}(y) = \lim_{N \rightarrow +\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x) (-q)^x$$

In particular, the  $q = 1$ -Volkenborn integral satisfies [1, eq. (1.6)]

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{+\infty} E_n(x) \frac{t^n}{n!}.$$

where  $E_n(x)$  is the Euler polynomial of degree  $n$ .

Another interesting case is the  $q = 0$ -Volkenborn integral which satisfies [5, p. 271]

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(x) \frac{t^n}{n!}$$

where  $B_n(x)$  is the Bernoulli polynomial of degree  $n$ . From this result we deduce the following

**Theorem.** *If  $f(x)$  is analytic in a neighborhood of 0 then its  $q = 0$ -Volkenborn integral can be computed as the expectation*<sup>1</sup>

$$(1.1) \quad \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \mathbb{E}f\left(x + \imath L_B - \frac{1}{2}\right)$$

where the random variable  $L_B$  follows the logistic distribution with density

$$(1.2) \quad \frac{\pi}{2} \operatorname{sech}^2(\pi x), \quad x \in \mathbb{R}.$$

Moreover, its  $q = 1$ -Volkenborn integral coincides with the expectation

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \mathbb{E}f\left(x + \imath L_E - \frac{1}{2}\right)$$

where the random variable  $L_E$  follows the hyperbolic secant distribution with density

$$(1.3) \quad \operatorname{sech}(\pi x), \quad x \in \mathbb{R}.$$

*Proof.* These results can be proved by computing the characteristic functions associated to the logistic distribution

$$\mathbb{E} \exp(\imath t L_B) = \frac{t}{2} \operatorname{csch}\left(\frac{t}{2}\right)$$

<sup>1</sup>in the rest of this paper, we use the notation  $\mathbb{E}_X f(X)$  for the probabilistic expectation  $\int f(x) p_X(x) dx$  where  $p_X$  is the probability density function of the random variable  $X$ . When no ambiguity occurs, we also denote  $\mathbb{E}f(X_1, X_2, \dots)$  the expectation over all random variables that appear as arguments of the function  $f$ .

and to the hyperbolic secant distribution

$$\mathbb{E} \exp(itL_E) = \operatorname{sech}\left(\frac{t}{2}\right).$$

□

The special case  $f(x) = x^n$  yields the following moment representation for the Bernoulli polynomial of degree  $n$

$$(1.4) \quad B_n(x) = \mathbb{E} \left( x + \imath L_B - \frac{1}{2} \right)^n$$

and

$$E_n(x) = \mathbb{E} \left( x + \imath L_E - \frac{1}{2} \right)^n$$

for the Euler polynomial of degree  $n$ . Moreover, choosing  $x = 0$  yields the following moment representation for the  $n$ -th Bernoulli number

$$B_n = B_n(0) = \mathbb{E} \left( \imath L_B - \frac{1}{2} \right)^n$$

and

$$(1.5) \quad E_n = E_n(0) = \mathbb{E} \left( \imath L_E - \frac{1}{2} \right)^n$$

for the  $n$ -th Euler number <sup>2</sup> where  $L_B$  and  $L_E$  follow respectively the logistic distribution (1.2) and the hyperbolic secant distribution (1.3).

An important feature of the logistic and hyperbolic secant random variables is the following cancellation property:

**Lemma 1.** *If  $U_B$  is a continuous random variable uniformly distributed over  $[0, 1]$  and independent of  $L_B$  then*

$$\mathbb{E} \left( x + \imath L_B - \frac{1}{2} + U_B \right)^n = x^n.$$

Accordingly, if  $U_E$  is a Rademacher distributed random variable ( $\Pr\{U_E = 0\} = \Pr\{U_E = 1\} = \frac{1}{2}$ ) then

$$\mathbb{E} \left( x + \imath L_E - \frac{1}{2} + U_E \right)^n = x^n.$$

Both results can be shown considering the characteristic functions of the involved variables; for example, in the Bernoulli case,

$$\mathbb{E} \exp \left( t \left( \imath L_B - \frac{1}{2} \right) \right) \exp(tU_B) = 1$$

so that all integer nonzero moments of the random variable  $\imath L_B - \frac{1}{2} + U_B$  vanish.

The Volkenborn integrals were used by Kim et al [1] to obtain non-trivial identities on Euler numbers  $E_n$  using integrals of the Bernstein polynomials defined as

$$\mathcal{B}_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq x \leq 1.$$

In this paper, we show that the probabilistic representation (1.1) of the Volkenborn integral makes its computation very easy to handle. We illustrate this fact by extending the non-trivial identities of [1] to the case of Bernoulli numbers. In the second section, we derive the probabilistic equivalent of the multidimensional Volkenborn integrals as introduced in [2] and we use it to prove a multivariate version of Raabe's multiplication theorem for Bernoulli and Euler polynomials.

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<sup>2</sup>note that it differs from the  $n$ -th Euler number of the first kind defined by  $\tilde{E}_n = 2^n E_n(\frac{1}{2})$ .

## 2. IDENTITIES FOR BERNOULLI NUMBERS AND POLYNOMIALS

**2.1. first-order identity.** In order to obtain non-trivial identities on Bernoulli numbers, we replace the Bernstein polynomials used by Kim et al by the Beta polynomials

$$\mathfrak{B}_{k,n}(x) = x^k (1+x)^{n-k}, \quad 0 \leq k \leq n$$

and, with  $X = \imath L_B - \frac{1}{2}$ , compute the expectation  $\mathbb{E}\mathfrak{B}_{k,n}(X)$  in two different ways:

- the first way is by applying the binomial formula

$$(2.1) \quad \mathbb{E}\mathfrak{B}_{k,n}(X) = \mathbb{E}X^k \sum_{j=0}^{n-k} \binom{n-k}{j} X^j = \sum_{j=0}^{n-k} \binom{n-k}{j} B_{j+k}$$

- the second way is by expressing  $X = (X+1) - 1$  so that

$$(2.2) \quad \mathbb{E}\mathfrak{B}_{n,k}(X) = \mathbb{E} \sum_{j=0}^k \binom{k}{j} (-1)^j (1+X)^{(n-k)+(k-j)} = \sum_{j=0}^k \binom{k}{j} (-1)^j \{B_{n-j} + \delta_{n-j-1}\};$$

since  $j \leq k \leq n$ , the Kronecker adds a term  $\binom{n-1}{n-1} (-1)^{n-1}$  if  $k = n-1$  or  $\binom{n}{n-1} (-1)^{n-1}$  if  $k = n$  and no term otherwise. We conclude the following

**Theorem 2.** *The Bernoulli numbers satisfy*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} B_{j+k} = \begin{cases} \sum_{j=0}^k \binom{k}{j} (-1)^j B_{n-j} & \text{if } 0 \leq k \leq n-2, \quad n \geq 2 \\ \sum_{j=0}^k \binom{k}{j} (-1)^j B_{n-j} + (-1)^{n-1} & \text{if } k = n-1, \quad n \geq 1 \\ \sum_{j=0}^k \binom{k}{j} (-1)^j B_{n-j} + n(-1)^{n-1} & \text{if } k = n, \quad n \geq 0 \end{cases}$$

We remark that the case  $k = n$  reads

$$B_n = \sum_{j=0}^n \binom{n}{j} (-1)^j B_{n-j} + n(-1)^{n-1}$$

**2.2. polynomial identities.** From (1.4), we deduce that the above results can be extended to the case of Bernoulli polynomials by choosing  $X = x + \imath L_B - \frac{1}{2}$ . We deduce the following

**Theorem 3.** *The Bernoulli polynomials satisfy, for all  $0 \leq k \leq n$ ,*

$$(2.3) \quad \sum_{j=0}^{n-k} \binom{n-k}{j} B_{j+k}(x) = \sum_{j=0}^k \binom{k}{j} (-1)^j B_{n-j}(x) + (nx - (n-k))x^{n-k-1}(x-1)^{k-1}$$

*Proof.* The left-hand side is a direct consequence of that of (2.1); the left-hand side of (2.2) with  $X = x + \imath L_B - \frac{1}{2}$  yields

$$\sum_{j=0}^k \binom{k}{j} (-1)^j B_{n-j}(x+1)$$

and since  $B_{n-j}(x+U) = x^{n-j}$ , we deduce  $B_{n-j}(x+1) - B_{n-j}(x) = (n-j)x^{n-j-1}$ . replacing in the above sum yields the result.  $\square$

We note that the case  $k = 0$  reads

$$\sum_{j=0}^n \binom{n}{j} B_j(x) = B_n(x) + nx^{n-1},$$

which can be restated as

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

and is nothing but the expression of the cancellation principle

$$\mathbb{E}B_{n-1}(x + U_B) = x^{n-1}.$$

## 3. A POLYNOMIAL EXTENSION TO KIM'S IDENTITY

In [1], the following identity is derived using the Bernstein polynomials  $\mathcal{B}_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}$

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{j+k} = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E_{n-j} + 2\delta_k.$$

We now provide the following polynomial extension of this identity

**Theorem 4.** *The Euler polynomials satisfy*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{j+k}(x) = (-1)^{n+k+1} \sum_{j=0}^k \binom{k}{j} E_{n-j}(x) + 2x^k (1-x)^{n-k}.$$

*Proof.* We start from the identity

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j x^{j+k} = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (1-x)^{n-j}$$

obtained by expanding either (left-hand side) the  $(1-x)^{n-k}$  term of the (right-hand side)  $x^k = (x-1+1)^k$  in the expression of the Bernstein polynomial. Replacing the variable  $x$  by  $x = X + \imath L_E - \frac{1}{2}$  and remarking that

$$\mathbb{E}(1-x)^{n-j} = \mathbb{E}\left(1 - X - \left(\imath L_E - \frac{1}{2}\right)\right)^{n-j} = (-1)^{n-j} E_{n-j}(X-1)$$

with, by the cancellation principle,

$$E_{n-j}(X-1) + E_{n-j}(X) = 2(X-1)^{n-j},$$

we deduce

$$\begin{aligned} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (1-x)^{n-j} &= \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (-1)^{n-j} \{-E_{n-j}(X) + 2(X-1)^{n-j}\} \\ &= (-1)^{n+k+1} \sum_{j=0}^k \binom{k}{j} E_{n-j}(X) + 2 \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} (1-X)^{n-j} \end{aligned}$$

this last sum being equal to  $2X^k(1-X)^{n-k}$ , hence the result.  $\square$

We notice that the case  $X = 0$  is

$$\sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{j+k} = (-1)^{n+k+1} \sum_{j=0}^k \binom{k}{j} E_{n-j} + 2\delta_k.$$

It can be shown that

$$(-1)^{n+k+1} \sum_{j=0}^k \binom{k}{j} E_{n-j} = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} E_{n-j}$$

as follows: using the moment representation (1.5) the right-hand side reads

$$\mathbb{E}\left(\imath L_E - \frac{1}{2}\right)^{n-k} \left(1 - \left(\imath L_E - \frac{1}{2}\right)\right)^k = \mathbb{E}\left(-\imath L_E - \frac{1}{2}\right)^{n-k} \left(1 - \left(-\imath L_E - \frac{1}{2}\right)\right)^k$$

by the symmetry of the hyperbolic secant distribution, and is thus equal to

$$(-1)^{n-k} \mathbb{E}\left(\imath L_E + \frac{1}{2}\right)^{n-k} \left(\imath L_E + \frac{3}{2}\right)^k = \mathbb{E}f\left(\left(\imath L_E - \frac{1}{2}\right) + 1\right)$$

with  $f(x) = (-1)^{n-k} x^{n-k} (x+1)^k$ . By the cancellation principle

$$\mathbb{E}f\left(\left(\imath L_E - \frac{1}{2}\right) + 1\right) + f\left(\imath L_E - \frac{1}{2}\right) = 2f(0) = 0$$

so that the right-hand side is equal to  $-\mathbb{E}f\left(\imath L_E - \frac{1}{2}\right)$  which coincides with the left-hand side; hence we recover Kim's identity.

## 4. MULTIDIMENSIONAL VOLKENBORN INTEGRAL

**4.1. Introduction.** In [2], a multivariate version of the Volkenborn integral is defined as

$$\int f(\mathbf{x}) d\mu_0(\mathbf{x}) = \int \dots \int f(x_1, \dots, x_k) d\mu_0(x_1) \dots d\mu_0(x_k).$$

In particular, it satisfies, with  $\mathbf{y} \in \mathbb{R}^k$  and the notation  $|\mathbf{y}| = \sum_{i=1}^k y_i$ ,

$$\int_{\mathbb{Z}_p^k} e^{(x+|\mathbf{y}|)t} d\mu_0(\mathbf{y}) = \left( \frac{t}{e^t - 1} \right)^k e^{xt}.$$

This multivariate version of the Volkenborn integral can again be expressed as an expectation over a simple random variable as shown now.

**4.2. Moment representation and elementary properties.** The Bernoulli polynomials  $B_n^{(k)}(x|\mathbf{a})$  of order  $k$  and degree  $n$  with  $x \in \mathbb{R}$  with parameter  $\mathbf{a} \in \mathbb{R}^k$ , also called Nörlund polynomials, are defined by the generating function [3, 1.13.1]

$$\sum_{n=0}^{+\infty} B_n^{(k)}(x|\mathbf{a}) \frac{t^n}{n!} = e^{xt} \prod_{j=1}^k \left( \frac{a_j t}{e^{a_j t} - 1} \right)$$

and the corresponding Bernoulli numbers  $B_n^{(k)}(\mathbf{a})$  by

$$B_n^{(k)}(\mathbf{a}) = B_n^{(k)}(0|\mathbf{a}) = \prod_{j=1}^k \left( \frac{a_j t}{e^{a_j t} - 1} \right).$$

In particular, taking  $a_j = 1$  for all  $j \in [1, k]$  and denoting

$$B_n^{(k)}(x) = B_n^{(k)}(x|1, 1, \dots, 1)$$

we deduce

$$\int_{\mathbb{Z}_p^k} e^{(x+|\mathbf{y}|)t} d\mu_0(\mathbf{y}) = \sum_{n=0}^{+\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

We provide a multidimensional extension of the moment representation (1.4) as follows

**Theorem 5.** *The Bernoulli polynomials  $B_n^{(k)}(x|\mathbf{a})$  satisfy*

$$(4.1) \quad B_n^{(k)}(x|\mathbf{a}) = \mathbb{E} \left( x + \sum_{j=1}^k a_j \left( \imath L_B^{(j)} - \frac{1}{2} \right) \right)^n$$

where the random variables  $\{L_B^{(j)}\}_{1 \leq j \leq k}$  are independent and follow the logistic distribution (1.2).

As a consequence, the Bernoulli numbers  $B_n^{(k)}(\mathbf{a})$  satisfy

$$(4.2) \quad B_n^{(k)}(\mathbf{a}) = \mathbb{E} \left( \sum_{j=1}^k a_j \left( \imath L_B^{(j)} - \frac{1}{2} \right) \right)^n$$

and the multivariate Volkenborn integral, with  $\mathbf{x} \in \mathbb{R}^k$ ,

$$\int_{\mathbb{Z}_p^k} f(|\mathbf{x}|) d\mu_0(\mathbf{x}) = \mathbb{E} f \left( \sum_{j=1}^k \left( x_j + \imath L_B^{(j)} - \frac{1}{2} \right) \right).$$

*Proof.* Let us compute the generating function

$$\sum_{n=0}^{+\infty} \mathbb{E} \left( x + \sum_{j=1}^k a_j \left( \imath L_B^{(j)} - \frac{1}{2} \right) \right)^n \frac{t^n}{n!} = \mathbb{E} \exp \left( tx + t \sum_{j=1}^k a_j \left( \imath L_B^{(j)} - \frac{1}{2} \right) \right) = e^{tx} \prod_{j=1}^k \mathbb{E} e^{ta_j (\imath L_B^{(j)} - \frac{1}{2})}$$

with, for a logistic distributed random variable  $L_j$ ,

$$\mathbb{E} e^{ta_j (\imath L_B^{(j)} - \frac{1}{2})} = \frac{a_j t}{e^{a_j t} - 1}$$

hence the result.  $\square$

The moment representation (4.1) allows to recover easily some well-known results about the higher-order Bernoulli polynomials.

**Proposition 6.** *The higher-order Bernoulli polynomials satisfy the identities*

$$(4.3) \quad B_n^{(1)}(x|a) = a^n B_n\left(\frac{x}{a}\right),$$

$$(4.4) \quad \sum_{l=0}^n \binom{n}{l} x^l B_{n-l}^{(k)}(y|\mathbf{a}) = B_n^{(k)}(x+y|\mathbf{a})$$

and

$$(4.5) \quad B_{2n+1}^{(k)}\left(\frac{a_1 + \dots + a_k}{2}|\mathbf{a}\right) = 0.$$

*Proof.* Identity (4.3) is a direct consequence of the moment representation (4.1); identity (4.4) is obtained using a binomial expansion of (4.1) and identity (4.5) by computing

$$B_{2n+1}^{(k)}\left(\frac{a_1 + \dots + a_k}{2}|\mathbf{a}\right) = \mathbb{E}\left(\iota \sum_{l=1}^k a_l L_B^{(l)}\right)^{2n+1}$$

and using the fact that the logistic density (1.2) is an even function.  $\square$

We also deduce straightforwardly from a multinomial expansion of the representations (4.1) and (4.2) the following

**Proposition 7.** *The higher-order Bernoulli polynomials satisfy*

$$B_n^{(k)}(x_1 + \dots + x_k|\mathbf{a}) = \sum_{i_1 + \dots + i_k = n} \binom{n}{i_1, \dots, i_k} B_{i_1}(x_1|a_1) \dots B_{i_k}(x_k|a_k)$$

and the higher-order Bernoulli numbers

$$B_n^{(k)}(\mathbf{a}) = \sum_{i_1 + \dots + i_k = n} \binom{n}{i_1, \dots, i_k} B_{i_1}(a_1) \dots B_{i_k}(a_k)$$

These results extend Corollary 5 and Corollary 6 in [2] which correspond to the case  $\mathbf{a} = (1, \dots, 1)$ .

**4.3. Kim's identity for Nörlund polynomials.** In order to highlight the efficiency of the moment representation (4.1), we derive now an extension of Kim's identity (2.3) to the case of Nörlund polynomials as follows.

**Theorem 8.** *For  $p \in \mathbb{N}$  and  $0 \leq k \leq n$ ,*

$$\sum_{j=0}^{n-k} \binom{n-k}{j} B_{j+k}^{(p)}(x) = \sum_{j=0}^k \binom{k}{j} (-1)^j \left\{ B_{n-j}^{(p)}(x) + (n-j) B_{n-j-1}^{(p-1)}(x) \right\}.$$

*Proof.* We start from the identity

$$(4.6) \quad \sum_{j=0}^{n-k} \binom{n-k}{j} X^{j+k} = \sum_{j=0}^k \binom{k}{j} (-1)^j (1+X)^{n-j}$$

and replace  $X$  by  $x + \sum_{l=1}^p \left(\iota L_B^{(l)} - \frac{1}{2}\right)$  so that, from 4.1, the left-hand side reads

$$\sum_{j=0}^{n-k} \binom{n-k}{j} B_{j+k}^{(p)}(x)$$

while the right-hand side is

$$\sum_{j=0}^k \binom{k}{j} (-1)^j B_{n-j}^{(p)}(x+1).$$

Since  $\mathbb{E}B_{n-j-1}^{(p)}(x+U) = B_{n-j-1}^{(p-1)}(x)$  with  $U$  uniform on  $[0, 1]$ , we deduce by the cancellation principle

$$\frac{B_{n-j}^{(p)}(x+1) - B_{n-j}^{(p)}(x)}{n-j} = B_{n-j-1}^{(p-1)}(x)$$

which yields the final result.  $\square$

**4.4. Kim's identity extended to multidimensional Euler polynomials.** We now provide a multidimensional version of the polynomial Kim identity derived in Theorem 4 as follows:

**Theorem 9.** *The multidimensional Euler polynomials satisfy the identity*

$$\begin{aligned} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j E_{j+k}^{(p)}(x) &= (-1)^{n+k+1} \sum_{j=0}^k \binom{k}{j} E_{n-j}^{(p)}(x) \\ &\quad + 2(-1)^{n+k} \sum_{j=0}^k \binom{k}{j} E_{n-j}^{(p-1)}(x-1) \end{aligned}$$

*Proof.* Starting again from identity (4.6), we take  $x = X + \sum_{l=1}^p \left( \imath L_E^{(l)} - \frac{1}{2} \right)$  and compute

$$(1-x)^{n-j} = \left( 1 - X - \sum_{l=1}^p \left( \imath L_E^{(l)} - \frac{1}{2} \right) \right)^{n-j} = (-1)^{n-j} E_{n-j}^{(p)}(X-1).$$

However, by the cancellation rule

$$E_{n-j}^{(p)}(X-1) + E_{n-j}^{(p)}(X) = 2E_{n-j}^{(p-1)}(X-1)$$

and the result follows.  $\square$

**4.5. Raabe's and Nielsen's multiplication theorem for Nörlund polynomials.** Raabe's usual multiplication theorem

$$m^{1-n} B_n(mx) = \sum_{l=0}^{m-1} B_n \left( x + \frac{l}{m} \right)$$

and

$$m^{-n} E_n(mx) = \sum_{l=0}^{m-1} (-1)^l E_n \left( x + \frac{l}{m} \right), \quad m \text{ odd}$$

and Nielsen's multiplication theorem

$$m^{-n} E_n(mx) = -\frac{2}{n+1} \sum_{l=0}^{m-1} (-1)^l B_{n+1} \left( x + \frac{l}{m} \right), \quad m \text{ even}$$

are an interesting feature of the Bernoulli polynomials since, as noted by Nielsen, [8, p. 54]

It is very curious, it seems to me, that there exist polynomials, with arbitrary degree, that satisfy equations of the above form. However, it is easy to prove that, up to an arbitrary constant factor, the  $B_n(x)$  and  $E_n(x)$  are the only polynomials that satisfy the mentioned property.

Using the moment representation and basic results from probability theory, we propose the following extension of Raabe's celebrated multiplication theorem to the multivariate case.

**Theorem 10.** *If  $m \in \mathbb{N}$ ,*

$$(4.7) \quad m^{k-n} B_n^{(k)}(mx|\mathbf{a}) = \sum_{l_1, \dots, l_k=0}^{m-1} B_n^{(k)} \left( x + \frac{1}{m} \sum_{i=1}^k a_i l_i | \mathbf{a} \right)$$

*and if moreover  $m$  is odd,*

$$(4.8) \quad m^{-n} E_n^{(k)}(mx|\mathbf{a}) = \sum_{l_1, \dots, l_k=0}^{m-1} (-1)^{l_1 + \dots + l_k} E_n^{(k)} \left( x + \frac{1}{m} \sum_{i=1}^k a_i l_i | \mathbf{a} \right).$$

*Proof.* Let us denote  $\{\tilde{U}^{(i)}\}_{1 \leq i \leq k}$  a set of  $k$  discrete random variables independent and uniformly distributed in the set  $\{0, \dots, m-1\}$  and  $\{U_B^{(i)}\}_{1 \leq i \leq k}$  a set of  $k$  continuous random variables independent and uniformly distributed on the interval  $[0, 1]$ . For the Bernoulli case, we have

$$\begin{aligned} \frac{1}{m^k} \sum_{l_1, \dots, l_k=0}^{m-1} B_n^{(k)} \left( x + \frac{1}{m} \sum_{i=1}^k a_i l_i | \mathbf{a} \right) &= \mathbb{E} \left( x + \sum_{i=1}^k a_i \left( \imath L_B^{(i)} - \frac{1}{2} \right) + \frac{1}{m} \sum_{i=1}^k a_i \tilde{U}^{(i)} \right)^n \\ &= \frac{1}{m^n} \mathbb{E} \left( mx + m \sum_{i=1}^k a_i \left( \imath L_B^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i \tilde{U}^{(i)} \right)^n \\ &= \frac{1}{m^n} \mathbb{E} \left( mx + m \sum_{i=1}^k a_i \left( \imath L_B^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i \tilde{U}^{(i)} + \sum_{i=1}^k a_i \left( \imath \tilde{L}_B^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i U_B^{(i)} \right)^n \end{aligned}$$

Now we use the fact that  $\tilde{U}^{(i)} + U_B^{(i)}$  has the same distribution as  $m U_B^{(i)}$  so that we obtain

$$\frac{1}{m^n} \mathbb{E} \left( mx + m \sum_{i=1}^k a_i \left( \imath L_B^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i \left( \imath \tilde{L}_B^{(i)} - \frac{1}{2} \right) + m \sum_{i=1}^k a_i U_B^{(i)} \right)^n$$

and applying the cancellation principle, we deduce

$$\frac{1}{m^n} \mathbb{E} \left( mx + \sum_{i=1}^k a_i \left( \imath \tilde{L}_B^{(i)} - \frac{1}{2} \right) \right)^n = \frac{1}{m^n} B_n^{(k)}(mx | \mathbf{a}).$$

For the Euler case, we need to use a signed measure (and then depart temporarily from the probabilistic context) defining the set  $\{\hat{U}^{(i)}\}_{1 \leq i \leq k}$  of  $k$  discrete variables independent such as each  $\hat{U}^{(i)}$  takes values in  $\{0, \dots, k, \dots, m-1\}$  with a weight  $(-1)^k$ . Then

$$\begin{aligned} \sum_{l_1, \dots, l_k=0}^{m-1} (-1)^{l_1 + \dots + l_k} E_n \left( x + \frac{1}{m} \sum_{i=1}^k a_i l_i | \mathbf{a} \right) &= \mathbb{E} \left( x + \sum_{i=1}^k a_i \left( \imath L_E^{(i)} - \frac{1}{2} \right) + \frac{1}{m} \sum_{i=1}^k a_i \hat{U}^{(i)} \right)^n \\ &= \frac{1}{m^n} \mathbb{E} \left( mx + m \sum_{i=1}^k a_i \left( \imath L_E^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i \hat{U}^{(i)} \right)^n \\ &= \frac{1}{m^n} \mathbb{E} \left( mx + m \sum_{i=1}^k a_i \left( \imath L_E^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i \hat{U}^{(i)} + \sum_{i=1}^k a_i \left( \imath \tilde{L}_E^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i U_E^{(i)} \right)^n \end{aligned}$$

where now  $\{U_E^{(i)}\}$  are independent Rademacher random variables and, since  $m$  is odd, each  $U_E^{(i)} + \hat{U}^{(i)}$  has the same distribution as  $m U_E^{(i)}$  so that we obtain

$$\frac{1}{m^n} \mathbb{E} \left( mx + m \sum_{i=1}^k a_i \left( \imath L_E^{(i)} - \frac{1}{2} \right) + \sum_{i=1}^k a_i \left( \imath \tilde{L}_E^{(i)} - \frac{1}{2} \right) + m \sum_{i=1}^k a_i U_E^{(i)} \right)^n$$

and from the cancellation principle, we deduce the result.  $\square$

Raabe's identity (4.7) and (4.8) are in fact given without proof in [7, eq. (1.6) and (1.7)]. The case for  $m$  even is not provided, so we prove now

**Theorem 11.** *With  $n \in \mathbb{N}$  and  $m$  even,*

$$m^{k-n} \left( -\frac{1}{2} \right)^k \frac{n!}{(n-k)!} \left( \prod_{i=1}^k a_i \right) E_{n-k}^{(k)}(mx) = \sum_{l_1, \dots, l_k=0}^{m-1} (-1)^{l_1 + \dots + l_k} B_n^{(k)} \left( x + \frac{1}{m} \sum_{i=1}^k a_i l_i | \mathbf{a} \right).$$



*Proof.* Let us define a variable  $W = \{0, \dots, m-1\}$  with weights  $(-1)^l$ . Then the right-hand side reads, with  $B_i = {}_iL_B^{(i)} - \frac{1}{2}$  and  $E_i = {}_iL_E^{(i)} - \frac{1}{2}$ ,

$$\begin{aligned} \mathbb{E} \left( x + \sum_{i=1}^k a_i \left( \frac{W_i}{m} + B_i \right) \right)^n &= m^{-n} \mathbb{E} \left( x + \sum_{i=1}^k a_i \left( W_i + mB_i + E_i + U_E^{(i)} \right) \right)^n \\ &= m^{-n} E_n \left( x + \sum_{i=1}^k a_i \left( W_i + mB_i + U_E^{(i)} \right) \right) \end{aligned}$$

but it can be checked that each  $W_i + U_E^{(i)}$  takes values 0 and  $m$  with respective weights  $\frac{1}{2}$  and  $-\frac{1}{2}$  so that we obtain

$$-\frac{1}{2}m^{-n} \left( E_n \left( x + m \sum_{i=1}^k a_i B_i + \sum_{i=1}^{k-1} (W_i + U_E^{(i)}) + ma_k \right) - E_n \left( x + m \sum_{i=1}^k a_i B_i + \sum_{i=1}^{k-1} (W_i + U_E^{(i)}) \right) \right)$$

which coincides with

$$-\frac{1}{2}m^{-n} E_n \left( x + m \sum_{i=1}^k a_i B_i + \sum_{i=1}^{k-1} (W_i + U_E^{(i)}) + ma_k U_B^{(k)} \right) = -\frac{1}{2}m^{-n} (ma_k) n E_{n-1} \left( x + \sum_{i=1}^{k-1} a_i (W_i + mB_i + U_E^{(i)}) \right)$$

since  $ma_k B_k$  and  $ma_k U_B^{(k)}$  cancel out. We are now back, up to a factor, to the same quantity as before except that  $n$  is replaced by  $n-1$  and  $k$  by  $k-1$ , hence the result.  $\square$

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